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TECHNICAL TRANSLATION

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THEORY OF THE MOTION OF A BODY WITH CAVITIES

PARTLY FILLED WITH A LIQUID

By D. E. Okhotsimskii

Translated from Prikladnaia Matematika i Mekhanika,
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THEORY OF THE MOTION OF A BODY WITH CAVITIES PARTLY FILLED WITH A LIQUID*

By D. E. Okhotsimskii

F 3 3

N. E. Zhukovskii [1] has given a detailed investigation of the problem of the motion of a rigid body containing cavities completely filled by a liquid. He showed that in the case of simply connected cavities the whole system can be replaced by a certain equivalent rigid body whose mass is equal to the mass of the system and which has a certain specified moment of inertia. This moment of inertia was calculated for a number of cavities of various shapes.

When the cavities are partially filled, there exists no equilvalent rigid body, although in certain special kinds of motion (where the force is applied by an impulse or in harmonic motion) it is possible to introduce inertial characteristics of the system analogous to the mass and moment of inertia, and to use these to set up the equations of motion and to analyze the behavior of the system under the action of external forces.

We give here the results of such an investigation for special kinds of cavities in the form of a circular cylinder and in the form of two concentric cylinders. The velocity potential is found. It is shown that there exist three independent inertial characteristics, and a method is given for calculating them. The concept of the center of inertia of the system is introduced, and a theorem analogous to Steiner's (parallel axis) theorem is proved. A method is given for setting up the equations of motion. An analysis and a physical explanation is given for the dependence of the inertial characteristics on the shape of the cavity and on the type of motion.

This article was written in 1950.** Since then, the following additions have been made: a cavity consisting of two concentric cylinders is treated, and a more complete investigation of the inertial characteristics is undertaken.

1. The determination of the velocity potential within the cavity between two circular cylinders. Consider a container, whose shape is that of the cavity between two concentric circular cylinders, filled with a massive ideal incompressible liquid (Fig. 1). Let h be the depth of the liquid, a the radius of the external cylinder, and ka the radius of the internal cylinder ($0 \leq k < 1$). Let the origin of the coordinate system be at the center of the free surface. Let the x

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and y axes lie in the horizontal plane and the z axis be directed vertically upward.

We shall restrict our treatment to linear considerations, and shall investigate the motion of the liquid which occurs if the container is subjected to motion in the vicinity of its initial position. This motion may, in particular, reduce to small oscillations about a stationary axis. In order to solve the problem of the general motion of the container, it is sufficient to be able to find the motion of the liquid if, for instance, the body undergoes rotation about the x axis and translation along the y axis.

We shall consider the xyz coordinate system to remain stationary. We shall call the motion in this stationary coordinate system absolute motion. We shall assume, further, that the absolute motion of the liquid is irrotational. If the motion started from rest, this follows from the ordinarily assumed absence of friction and characteristics of mass forces.

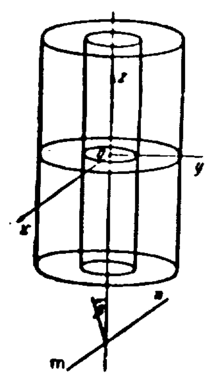

Fig. 1

Let φ be the absolute velocity potential. The condition of incompressibility gives

$$\Delta\varphi = 0 . \tag{1.1}$$

The boundary conditions on the external and internal walls of the container are

$$\left[\frac{\partial\varphi}{\partial r}\right]_{r=a} = v_{r1}, \qquad \left[\frac{\partial\varphi}{\partial r}\right]_{r=ka} = v_{r2} \tag{1.2}$$

and at the bottom they are

$$\left[\frac{\partial\varphi}{\partial z}\right]_{z=-h} = v_z \tag{1.3}$$

where v_{r1}, v_{r2} and v_z are the velocity components of points on the boundary surface in the direction of the normal to the initial position of the surface.

Throughout the whole liquid, we have the integral

$$\frac{p-p_0}{\rho} = -\frac{\partial\varphi}{\partial t} - gz \tag{1.4}$$

where ρ is the liquid density, p is the pressure, p_0 s the pressure on the free surface, and g is the intensity of the mass force acting along the negative z axis. In particular, this may be the acceleration of gravity. Equation (1.4) gives the condition

$$\left[\frac{\partial\varphi}{\partial t}\right]_{z=0} + g\zeta = 0 \tag{1.5}$$

on the free surface where $\zeta(x,y)$ is the equation of the free surface.

The quantity ζ may be considered the displacement along the z axis of a particle which initially lies in the z = 0 plane. We shall attempt to find the potential in the form of the sum of two potentials

$$\varphi = \varphi_1 + \varphi_2 \,. \qquad (1.6)$$

Let us consider the same cavity bounded at the top by a plane surface whose distance from the free surface is equal to the depth of the liquid, and let us assume that this cavity is completely filled by liquid. We shall consider φ_1 to be the absolute velocity potential for this case. In the lower part of the cavity this velocity potential will satisfy the above conditions at the walls and at the bottom, but will not, in general satisfy the condition at z = 0. We shall choose F 3 3 φ_2 such that it vanish at the walls and at the bottom, and such that its sum with φ_1 satisfy the condition at z = 0.

This potential φ_2 gives the wave motion that would take place in the liquid if the container were to remain stationary and the liquid were acted upon by the nonuniformities in the pressure on the z = 0 plane that are due to the motion with the potential φ_1.

Like Zhukovskii [1], we shall attempt to find φ_1 in the form

$$\varphi_1 = \dot{\beta}(F - yz) + v_0 y \qquad (1.7)$$

where $\dot{\beta}$ is the angular velocity of rotation of the container about the x axis, and v_0 is the velocity of the center of the free surface as it moves together with the container in the y direction.

The function F(x,y,z) must satisfy Laplace's equation •

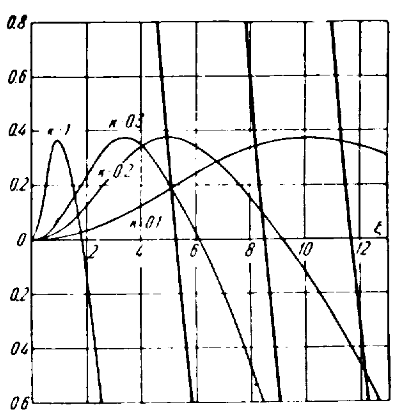

Fig. 2.

$$\Delta F = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 F}{\partial z^2} = 0 \qquad (1.8)$$

$$\left(r = \sqrt{x^2 + y^2},\ \eta = \operatorname{arc\,tg}\frac{y}{x}\right) \qquad (1.8)$$

with the boundary conditions

$$\frac{\partial F}{\partial r} = 0 \qquad \text{for } r = a,\ r = ka \qquad (1.9)$$

at the external and internal walls, and

$$\frac{\partial F}{\partial z} = 2y \qquad \text{for } z = \pm h \qquad (1.10)$$

at the top and bottom of the container. We wish to find a solution in the form of a linear combination of terms such as

$$F = Z(z)\, H(\eta)\, R(r) \qquad (1.11)$$

• In this paper 'tg' ≡ 'tan', 'th' ≡ 'tanh', 'sh' ≡ 'sinh', 'ch' ≡ 'cosh' – Publisher.

Separating variables, we obtain the following equations:

$$\frac{d^2Z}{dz^2} - \lambda^2 Z = 0, \quad \frac{d^2H}{d\eta^2} + m^2 H = 0, \quad \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{m^2}{r^2}\right)R = 0 \tag{1.12}$$

where $\lambda^2 > 0$. From the condition that $Z(z)$ is an even function we have

$$Z(z) = \mathrm{sh}(\lambda z) \tag{1.13}$$

Also

$$H(\eta) = \sin(m\eta + \varepsilon), \qquad R(r) = J_m(\lambda r) + \gamma N_m(\lambda r) \tag{1.14}$$

Because $H(\eta)$ is periodic, the number m must be integral. The quantities ϵ and γ are constants, and J_m and N_m are Bessel functions of the first and second kind of order m. To determine the function F we take $m = 1$, and $\epsilon = 0$. The constants λ and γ are determined from the boundary conditions at the walls of the container. We use the notation

$$f(\zeta) = \frac{J_1'(\zeta)}{N_1'(\zeta)} \tag{1.15}$$

where the primes denote differentiation with respect to ζ. We seek the roots of the equation

$$f(\zeta) = f(k\zeta) \tag{1.16}$$

The function $f(\zeta)$ is shown in Fig. 2 ($k = 1$). Also shown is $f(k\zeta)$ for $k = 0.1$, $k = 0.2$ and $k = 0.3$. When $k = 0$ the curve runs along the abscissa axis. Eq. (1.16) has an infinite number of roots ζ_n. The corresponding values for γ_n are

$$\gamma_n = -f(\zeta_n) \tag{1.17}$$

In the table are shown the values of the roots ζ_n and the quantities γ_n and $n = 1 - 4$ and $k = 0, 0.1, 0.2, 0.3$.

	n	ζ_n	γ_n	a_n	b_n	d_n
$k = 0$	1	1.8412	—	1.718	—	0.4184
	2	5.331	—	−2.889	—	0.03648
	3	8.536	0	3.659	0	0.01391
	4	11.71	—	−4.286	—	0.00734
	5	14.86	—	4.831	—	0.00446
$k = 0.1$	1	1.8036	−0.02562	1.703	−0.04362	0.4126
	2	5.137	−0.1902	−2.786	0.5298	0.04251
	3	8.199	−0.3460	3.389	−1.1725	0.01117
	4	11.36	−0.3581	−3.976	1.4239	0.01024
	5	14.63	−0.2366	4.654	−1.1035	0.00342
$k = 0,2$	1	1.7053	−0.09038	1.657	−0.1498	0.3995
	2	4.962	−0.3728	−2.611	0.9734	0.05837
	3	8.433	−0.1023	3.617	−0.3700	0.00916
	4	12.16	+0.4844	−3.932	−1.9048	0.01184
$k = 0.3$	1	1.582	−0.1665	1.593	−0.2652	0.3809
	2	5.138	−0.1893	−2.736	0.5273	0.0727
	3	9.307	+0.9544	2.765	2.6387	0.00718
	4	13.69	−2.350	1.815	−4.2650	0.01160

The functions R_n which correspond to the roots of Eq. (1.16) may be conveniently normalized, for example, by setting their values at $r = a$ equal to unity. We obtain

$$R_n\left(\frac{r}{a}\right) = a_n J_1\left(\zeta_n \frac{r}{a}\right) + b_n N_1\left(\zeta_n \frac{r}{a}\right)$$
$$a_n = \frac{1}{J_1(\zeta_n) + \gamma_n N_1(\zeta_n)}, \qquad b_n = a_n \gamma_n \tag{1.18}$$

In the special case k = 0, we have $\gamma_n = 0$ and

$$R_n\left(\frac{r}{a}\right) = \frac{J_1(\zeta_n r/a)}{J_1(\zeta_n)} \tag{1.19}$$

Values of a_n and b_n are given in the Table.

Figures 3, 4 and 5 show graphs of R_n for n = 1,2 and 3 and for k = 0, 0.1, 0.2 and 0.3. We see that functions corresponding to a given n and various k do not differ greatly, the only differences between them occurring essentially in the first half wave from the center.

F 3 3

Let us write F in the form

$$F = \sin\eta \sum_{n=1}^{\infty} C_n \,\mathrm{sh}\left(\zeta_n \frac{z}{a}\right) R_n\left(\frac{r}{a}\right) \tag{1.20}$$

The coefficients C_n can be chosen from the boundary conditions (1.10) at the upper and lower surfaces. Expanding y in a series of R_n functions, we have

$$y = 2a\sin\eta \sum_{n=1}^{\infty} d_n R_n\left(\frac{r}{a}\right) \qquad d_n = \frac{1 - kR_n(k)}{(\zeta_n^2 - 1) - (k^2\zeta_n^2 - 1)R_n^2(k)}\,. \tag{1.21}$$

In the special case k = 0, we have

$$d_n = \frac{1}{\zeta_n^2 - 1}, \qquad y = 2a\sin\eta \sum_{n=1}^{\infty} \frac{1}{\zeta_n^2 - 1}\,\frac{J_1(\zeta_n r/a)}{J_1(\zeta_n)}\,. \tag{1.22}$$

The d_n are given in the Table.

Using the boundary condition, we arrive at

$$F = 4a^2 \sin\eta \sum_{n=1}^{\infty} \frac{d_n}{\zeta_n}\,\frac{\mathrm{sh}(\zeta_n z/a)}{\mathrm{ch}(\zeta_n h/a)} R_n\left(\frac{r}{a}\right). \tag{1.23}$$

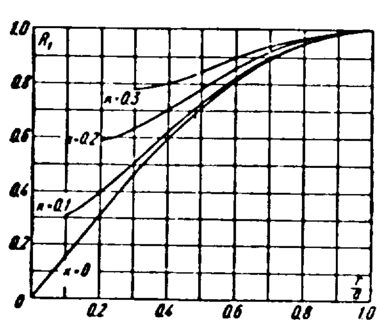

Fig. 3

Let us now find the potential φ_2 for the wave motion. This is a harmonic function satisfying the condition $\partial\varphi_2/\partial r = 0$ at the walls and $\partial\varphi_2/\partial z = 0$ at the bottom.

We shall attempt to find a solution in the form of the sum

$$\varphi_2 = f(t)\,Z(z)\,H(\eta)\,R(r) \tag{1.24}$$

where f(t) is a function of time. It is clear that in investigating forced wave motion which arises in the plane motion of the container, one may choose $H(\eta)$

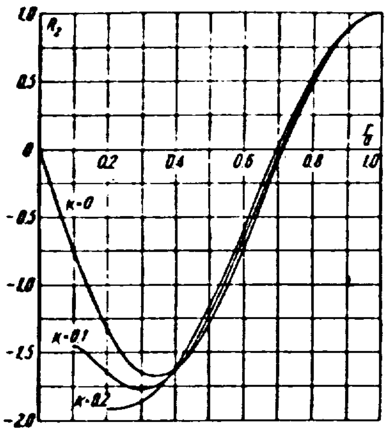

Fig. 4.

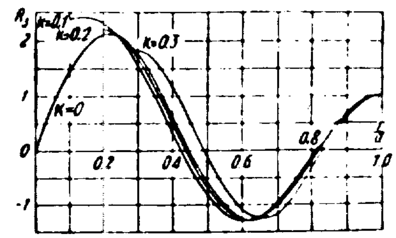

Fig. 5.

and R(r) of the same form as their choice in F.

Figures 3,4 and 5 show the form of the waves for the first three harmonics. As for the $\underline{z}$ dependence, we satisfy the condition at the bottom by setting

$$Z(z) = \mathrm{ch}\left(\zeta_n \frac{z+h}{a}\right). \tag{1.25}$$

For the purpose of simplifying the equations for the time dependent functions, let us write φ_2 in the form

$$\varphi_2 = 2a^2 \sin\eta \sum_{n=1}^{\infty} \dot{\chi}_n(t) \frac{\mathrm{ch}\left(\zeta_n \frac{z+h}{a}\right)}{\mathrm{ch}\left(\zeta_n \frac{h}{a}\right)} d_n R_n\left(\frac{r}{a}\right) \tag{1.26}$$

where the $\chi_n(t)$ are undetermined functions of the time. The dot denotes differentiation with respect to time. We may rewrite the condition at the surface in the form

$$\frac{\partial\varphi_2}{\partial t}\Big|_{z=0} + g\zeta_2 = -\left(\frac{\partial\varphi_1}{\partial t}\Big|_{z=0} + g\zeta\right) \tag{1.27}$$

where ζ_1 and ζ_2 are the displacements due to motion with the potentials φ_1 and φ_2, so that

$$\zeta_1 = \int_0^t \frac{\partial\varphi_1}{\partial z}\Big|_{z=0} dt, \qquad \zeta_2 = \int_0^t \frac{\partial\varphi_2}{\partial z}\Big|_{z=0} dt$$

We calculate the left and right sides of (1.27) and equate coefficients, obtaining the following equations for the $\chi_n(t)$:

$$\ddot{\chi}_n + \omega_n^2\chi_n = -\frac{\dot{r}_0}{a} + \frac{g}{a}\left(1 - \frac{2}{\mathrm{ch}\,(\zeta_n h/a)}\right)\beta, \qquad \omega_n^2 = \zeta_n \frac{g}{a}\,\mathrm{th}\left(\zeta_n \frac{h}{a}\right). \tag{1.28}$$

The final expression for the absolute velocity potential of the liquid will be

$$\varphi = 2a^2 \sin\eta \sum_{n=1}^{\infty} \left\{\frac{\mathrm{ch}\left(\zeta_n \frac{z+h}{a}\right)}{\mathrm{ch}\left(\zeta_n \frac{h}{a}\right)}\dot{\chi}_n + \left(\frac{2}{\zeta_n}\frac{\mathrm{sh}\left(\zeta_n \frac{z}{a}\right)}{\mathrm{ch}\left(\zeta_n \frac{h}{a}\right)} - \frac{z}{a}\right)\dot{\beta} + \frac{r_0}{a}\right\} d_n R_n\left(\frac{r}{a}\right) \tag{1.29}$$

We note that solutions of the form of (1.24) can be used to construct potentials, combinations of which can represent arbitrary wave motion of the liquid in the container of the given shape. These potentials are of the form

$$\varphi^* = F(t) {}^{\sin}_{\cos} (m\eta) \operatorname{ch}\left(\zeta_n{}^m \frac{z+h}{a}\right) R_n{}^m\left(\frac{r}{a}\right) \qquad \begin{pmatrix} m = 0, 1, 2, \dots \\ n = 1, 2, 3, \dots \end{pmatrix}. \tag{1.30}$$

Here

$$R_n{}^m = \frac{J_m(\zeta_n{}^m r/a) + \gamma_n{}^m N_m(\zeta_n{}^m r/a)}{J_m(\zeta_n{}^m) + \gamma_n{}^m N_m(\zeta_n{}^m)} \tag{1.31}$$

In the above, the $\zeta_n{}^m$ are roots of the equation

$$\frac{J_m{}'(\zeta)}{N_m{}'(\zeta)} = \frac{J_m{}'(k\zeta)}{N_m{}'(k\zeta)} \tag{1.32}$$

and the $\gamma_n{}^m$ are the ratios of the derivatives when $\zeta = \zeta_n{}^m$.

In the case we are considering, that of plane motion of the container, all the potentials except those which enter into the expression for φ_2 satisfy a homogeneous equation of the form

$$\ddot{F} + \zeta_n{}^m \frac{g}{a} \operatorname{th}\left(\zeta_n{}^m \frac{h}{a}\right) F = 0 \tag{1.33}$$

in time, and correspond to free vibrations of the liquid. If any one of these is added to φ it will alter neither the boundary conditions nor the conditions on the free surface.

If the liquid started from rest, the free-vibration potentials vanish identically. We then also have

$$\chi_n = \dot{\chi}_n = 0 \qquad \text{for} \quad t = 0 \tag{1.34}$$

These conditions will be the initial conditions for Equations (1.28).

2. Reactions at the walls. Effective moments of inertia and effective masses.

In rotation about a stationary axis and in translational motion, the inertia of an absolutely rigid body is characterized by the moment of inertia I and the mass m. The "resistance" of a body to these two types of motion is given by

$$-I\ddot{\beta}, \ -m\ddot{\gamma} \tag{2.1}$$

where $\ddot{\beta}$ and $\ddot{\gamma}$ are the angular and linear accelerations.

One of the most important results obtained by N. E. Zhukovskii [1] is that if a body contains simply connected cavities completely filled by a liquid, it can be replaced by an equivalent rigid body. Under the influence of external forces the original body and the equivalent rigid body will move in exactly the same way. The "resistance" of such a system will therefore be given by Equations (2.1), where I will be the moment of inertia of the equivalent rigid body. The mass of the equivalent body is found to be the same as the mass of the original system.

For motion of a body whose cavities are only partly filled by a liquid, the situation is different. The pressure at any point of the liquid is given by

$$p - p_0 = -\rho\left(\frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_2}{\partial t}\right) - \rho g z \quad (2.2)$$

The right side of this equation contains, in addition to the hydrostatic term, the quantities $\partial\varphi_1/\partial t$ and $\partial\varphi_2/\partial t$. The first of these is proportional to $\ddot{\beta}$, but the second is in general not proportional to this quantity.

Thus when there is a free surface it is impossible to introduce an equivalent rigid body. The inertia of such a system to rotation about a stationary axis and to translational motion cannot, in general, be reduced to quantities analogous to moment of inertia and mass which are independent both of time and of the type of motion.

Nevertheless, even if there is a free surface, there exist certain definite special cases of motion in which the reaction of the liquid on the walls is proportional to the acceleration of the container. It is then possible to introduce external inertial characteristics of the system, such as effective moments of inertia and effective masses, which do not depend on the time, but which exist, unfortunately, only for the given special type of motion.

F 3 3

We shall here consider two such special cases, namely the beginning of motion from a state of rest and steady state harmonic vibrations of the container.

We shall calculate the resultant force and the moment of the pressure forces exerted by the liquid on the walls of the container when the container rotates about the x axis and when it translates in the y direction. The formulas we obtain can be used to calculate the forces and moments for general space motion.

We shall calculate the forces, moments, and inertial characteristics for the case $k = 0$, which is the most interesting case of a container in the shape of a circular cylinder. The formulas we have obtained above for the velocity potential can be used to perform a similar calculation for any case in which $k \neq 0$. Many of the conclusions reached in the case of a cylindrical cavity are valid also for the more general case.

Let us consider that the hydrostatic pressure forces, which occur when the cylinder rotates about the x axis, are acting at the origin. For the resultant we have $Q_x = Q_y = 0$, $Q_z = -mg$, and for the resultant moment $L_y = L_z = 0$; the third component L_x of the moment can be calculated directly, but it is simpler to proceed differently.

In our case the pressure forces on the container will be equal and opposite to the pressure forces on the container upon immersion in the liquid.

For a circular cylinder immersed vertically, however, it is known that there exists a metacenter at a distance

$$d = \frac{I_x}{V} = \frac{a^2}{4h}$$

above the center of the cylinder, where $I_x = {}^1/_4\,\pi a^4$ is the moment of inertia of the cross sectional area about the $\underline{x}$ axis, $V = \pi a^2 h$ is the volume of the displaced liquid, and $\underline{h}$ is the depth to which the cylinder is immersed.

From this it follows that the moment of the hydrostatic forces about the $\underline{x}$ axis is

$$L_x = -mg\left(\frac{h}{2} - \frac{a^2}{4h}\right)\beta \qquad (2.3)$$

F 3 3

Rotating about a horizontal axis a distance L below the $\underline{x}$ axis, we have

$$L_x = mg\left(L - \frac{h}{2} + \frac{a^2}{4h}\right)\beta \qquad (2.4)$$

Going on to a container in motion, it is convenient to consider Q_z and L_x as external. We shall consider only the first term on the right side of (2.2) in calculating the inertial characteristics. This term gives the additional pressure which arises when the container is in motion. The components of the resultant and of the moment will be

$$P_x = P_z = 0, \qquad M_y = M_z = 0.$$

The other components, P_y and M_x are given by

$$P_y = -a\rho \int_{-h}^{0}\int_{0}^{2\pi}\left(\frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_2}{\partial t}\right)\sin\eta\, d\eta\, dz \qquad (2.5)$$

$$M_x = a\rho \int_{-h}^{0}\int_{0}^{2\pi}\left(\frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_2}{\partial t}\right) z\sin\eta\, d\eta\, dz + \rho\int_{0}^{a}\int_{0}^{2\pi}\left(\frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_2}{\partial t}\right) r^2\sin\eta\, d\eta\, dr. \qquad (2.6)$$

In the equation for P_y the integral is taken over the side walls of the original cylinder. In the equation for M_x the first integral is taken over the side walls, and the second over the bottom of the original cylinder.

By analogy with the case of a solidified liquid, we may write for rotation about the $\underline{x}$ axis

$$P_y = -m_0 \frac{1}{2} h\ddot{\beta}, \qquad M_x = -I_0\ddot{\beta} \qquad (2.7)$$

where m_0 and I_0 have the dimensions of mass and moment of inertia, respectively.

For motion of any arbitrary type these quantities will, in general, be variable. In our case we have

$$\frac{\partial\varphi_1}{\partial t} = \ddot{\beta}(F - zy) \qquad \left(F = 4a^2\sin\eta \sum_{n=1}^{\infty} \frac{1}{\zeta_n(\zeta_n^2 - 1)} \frac{J_1(\zeta_n r/a)\,\mathrm{sh}(\zeta_n z/a)}{J_1(\zeta_n)\,\mathrm{ch}(\zeta_n h/a)}\right). \qquad (2.8)$$

Further,

$$\frac{\partial \varphi_2}{\partial t} = 2a^2 \sin\eta \sum_{n=1}^{\infty} \frac{\ddot{\chi}_n J_1(\zeta_n r/a)\,\mathrm{ch}\,[\zeta_n(z+h)/a]}{(\zeta_n^2-1)\,J_1(\zeta_n)\,\mathrm{ch}\,(\zeta_n h/a)} \tag{2.9}$$

where the $\chi_n(t)$ satisfy the equations

$$\ddot{\chi}_n + \omega_n^2\chi_n = \frac{g}{a}\left(1-\frac{2}{\mathrm{ch}\,\mu_n}\right)\ddot{\beta} \tag{2.10}$$

The quantities m_0 and I_0 are then given by

$$m_0 = m\left\{1-\frac{1}{h^{\circ 2}}\left(1-8\sum_{n=1}^{\infty}\frac{\varepsilon_n}{\mathrm{ch}\,\mu_n}\right)+\frac{4}{h^{\circ 2}}\sum\frac{\ddot{\chi}_n}{\ddot{\beta}}\,\varepsilon_n\zeta_n\,\mathrm{th}\,\mu_n\right\} \tag{2.11}$$

$$I_0 = ma^2\left\{\left(\frac{1}{4}+\frac{h^{\circ 2}}{3}\right)-\left(1-\frac{8}{h^\circ}\sum_{n=1}^{\infty}\frac{\varepsilon_n\,\mathrm{th}\,\mu_n}{\zeta_n}\right)+\frac{2}{h^\circ}\sum_{n=1}^{\infty}\frac{\ddot{\chi}_n}{\ddot{\beta}}\,\varepsilon_n\left(1-\frac{2}{\mathrm{ch}\,\mu_n}\right)\right\} \tag{2.12}$$

F 3 3

where m is the mass of the liquid and we have written

$$h^\circ = \frac{h}{a}, \qquad \mu_n = \zeta_n\frac{h}{a}, \qquad \varepsilon_n = \frac{1}{\zeta_n^2(\zeta_n^2-1)}$$

and where $\ddot{\chi}_n$ is found by solving (2.10).

In arriving at Equations (2.11) and (2.12), we used the formula

$$\int_0^a J_1\left(\zeta_n\frac{r}{a}\right)r^2\,dr = a^3\frac{J_1(\zeta_n)}{\zeta_n^2}$$

and the fact that the numbers ζ_n satisfy the relations

$$\sum_{n=1}^{\infty}\frac{1}{\zeta_n^2-1}=\frac{1}{2}, \qquad \sum_{n=1}^{\infty}\frac{1}{\zeta_n^2(\zeta_n^2-1)}=\frac{1}{8}$$

which are easily derived.

Let us now consider special cases. If the liquid were to solidify, we would have

$$m_0 = m, \qquad I_0 = ma^2\left(\tfrac{1}{4}+\tfrac{1}{3}h^{\circ 2}\right). \tag{2.13}$$

Thus the first terms of (2.11) and (2.12) are those for a solidified liquid.

If the motion is just beginning or if the force is applied by a sudden impulse, Equation (2.10) and the initial conditions give $\ddot{\chi} = 0$ and only the first two terms in the equations for m_0 and I_0. We then have

$$m_0 = m\left\{1-\frac{1}{h^{\circ 2}}\left(1-8\sum_{n=1}^{\infty}\frac{\varepsilon_n}{\mathrm{ch}\,\mu_n}\right)\right\} \tag{2.14}$$

$$I_0 = ma^2\left\{\left(\frac{1}{4}+\frac{h^{\circ 2}}{3}\right)-\left(1-\frac{8}{h^\circ}\sum_{n=1}^{\infty}\frac{\varepsilon_n\,\mathrm{th}\,\mu_n}{\zeta_n}\right)\right\}. \tag{2.15}$$

Finally, in the case of simple harmonic motion Equation (2.10) gives

$$\frac{\ddot{\chi}_n}{\ddot{\beta}} = \frac{g}{a}\,\frac{1}{\omega_n^2-\omega^2}\left(1-\frac{2}{\mathrm{ch}\,\mu_n}\right)$$

We introduce the dimensionless parameter α given by

$$\alpha = \frac{a\omega^2}{g} \tag{2.16}$$

This parameter gives the ratio of the square of the forced vibration frequency to the square of the oscillation frequency of a pendulum of length $\underline{a}$. We have

$$\frac{\ddot{\chi}_n}{\ddot{\beta}} = \frac{1}{\zeta_n \operatorname{th} \mu_n - \alpha}\left(1 - \frac{2}{\operatorname{ch} \mu_n}\right).$$

Inserting this into (2.11) and (2.12) and performing the necessary operations, we obtain

$$m_0 = m\left\{1 - \frac{1}{2h^{\circ 2}}\left[1 - 8\sum_{n=1}^{\infty} \frac{\alpha\varepsilon_n}{\zeta_n \operatorname{th} \mu_n - \alpha}\left(1 - \frac{2}{\operatorname{ch} \mu_n}\right)\right]\right. \tag{2.17}$$

F 3 3

$$I_0 = ma^2\left\{\frac{h^{\circ 2}}{3} - \frac{3}{4} + \frac{8}{h^\circ}\sum_{n=1}^{\infty} \frac{\varepsilon_n \operatorname{th} \mu_n}{\zeta_n} + \frac{2}{h^\circ}\sum_{n=1}^{\infty} \frac{\varepsilon_n}{\zeta_n \operatorname{th} \mu_n - \alpha}\left(1 - \frac{2}{\operatorname{ch} \mu_n}\right)^2\right\}. \tag{2.18}$$

Let us now calculate the forces and moments for translation in the $\underline{y}$ direction. Let us write them in the form

$$P_y = -m_1\ddot{\gamma}, \qquad M_x = -m_2\frac{h}{2}\ddot{\gamma}. \tag{2.19}$$

We have

$$\frac{\partial \varphi_1}{\partial t} = \ddot{\gamma} y . \tag{2.20}$$

The derivative $\partial\varphi_2/\partial t$ is given by (2.9), and the χ_n must satisfy the equations

$$\ddot{\chi}_n + \omega_n^2 \chi_n = -\frac{\ddot{\gamma}}{a} . \tag{2.21}$$

Inserting (2.19) into (2.5) and (2.6) we arrive at

$$m_1 = m\left\{1 + \frac{2}{h^\circ}\sum_{n=1}^{\infty} \frac{a\ddot{\chi}_n}{\ddot{\gamma}}\varepsilon_n \zeta_n \operatorname{th} \mu_n\right\} \tag{2.22}$$

$$m_2 = m\left\{1 - \frac{1}{2h^{\circ 2}}\left[1 - 8\sum_{k=1}^{\infty} \frac{a\ddot{\chi}_n}{\ddot{\gamma}}\varepsilon_n\left(1 - \frac{2}{\operatorname{ch} \mu_n}\right)\right]\right\} \tag{2.23}$$

where $\ddot{\chi}_n$ is obtained from a solution of (2.21).

Let us now consider special cases. For a solidified liquid we have $m_1 = m_2 = m$. If motion is just beginning and if the forces are applied impulsively, we have

$$m_1 = m\left\{1 - \frac{2}{h^\circ}\sum_{n=1}^{\infty} \varepsilon_n \zeta_n \operatorname{th} \mu_n\right\} \tag{2.24}$$

$$m_2 = m\left\{1 - \frac{1}{h^{\circ 2}}\left(1 - 8\sum_{n=1}^{\infty} \frac{\varepsilon_n}{\operatorname{ch} \mu_n}\right)\right\} . \tag{2.25}$$

For steady state harmonic motion we have

$$\frac{\ddot{\chi}_n a}{\ddot{\gamma}} = \frac{\alpha}{\zeta_n \operatorname{th} \mu_n - \alpha} .$$

The formulas for m_1 and m_2 then become

$$m_1 = m\left\{1 + \frac{2}{h^\circ}\sum_{n=1}^{\infty}\frac{\alpha\zeta_n \operatorname{th}\mu_n}{\zeta_n \operatorname{th}\mu_n - \alpha}\varepsilon_n\right\} \tag{2.26}$$

$$m_2 = m\left\{1 - \frac{1}{2h^{\circ 2}}\left[1 - 8\sum_{n=1}^{\infty}\frac{\alpha\varepsilon_n}{\zeta_n \operatorname{th}\mu_n - \alpha}\left(1 - \frac{2}{\operatorname{ch}\mu_n}\right)\right]\right\}. \tag{2.27}$$

A comparison of the formulas obtained shows that the inertial characteristics for the start of motion can be obtained from the corresponding quantities for harmonic oscillations of the container and by going to the limit $\alpha \to \infty$. The parameter α is the Frud* number for our problem. We see that the limit $\alpha \to \infty$ is obtained when $\omega \to \infty$, when $a \to \infty$, or when $g \to 0$. If $g = 0$, this means that the motion takes place in the absence of mass forces. When impulsive forces act on the container, finite mass forces may be neglected, and the motion takes place as though it were true that $g = 0$.

Comparing (2.14) and (2.17) with (2.25) and (2.27), we see that for all values of α, including the limit $\alpha \to \infty$, we have identically

$$m_2 = m_0. \tag{2.28}$$

This means that in the cases we are considering, in which there exist time-independent inertial characteristics, there are not four, but three independent characteristics. These are the effective moment of inertia I_0 and the effective masses m_0 and m_1.

<u>3. Center of inertia. Steiner's theorem. The equations of motion of the system with a liquid filling.</u> Let us consider the motion of the container about an axis mn passing through the cylindrical axis and parallel to the $\underline{x}$ axis (Fig. 1). Let β be the angular displacement, and γ the displacement of the origin in the $\underline{y}$ direction. According to Equations (2.7) and (2.19) P_y and M_x are given by

$$P_y = -m_1\ddot{\gamma} - \frac{1}{2}hm_0\ddot{\beta}. \tag{3.1}$$

$$M_x = -\frac{1}{2}hm_0\ddot{\gamma} - I_0\ddot{\beta}. \tag{3.2}$$

We have set $m_2 = m_0$, since we shall assume either that motion is just beginning or that it is harmonic. Let L be the distance from the $\underline{x}$ axis to the axis mn (Fig. 1). We then have

$$\gamma = -\beta L, \qquad P_y = \left(m_1 L - \frac{1}{2}hm_0\right)\ddot{\beta}. \tag{3.3}$$

Let us choose the position of the axis of rotation so that the resultant of the pressure forces vanish, the set of these forces reducing to a couple.

Setting $P_y = 0$ in (3.3), we find that the distance from the $\underline{x}$ axis to the axis

* Transliteration of Russian — Publisher's note.

of rotation must be

$$L_C = \frac{h}{2} \frac{m_0}{m_1} . \qquad (3.4)$$

Let us call the point at which this position of the mn axis intersects the cylindrical axis the center of inertia of the liquid in the cylindrical container.

According to (3.2), (3.3) and (3.4) the moment of the pressure forces which is obtained when the system rotates about an axis passing through the center of inertia is

$$M_C = -I_C \ddot{\beta}, \qquad I_c = I_0 - m_1 L_C^2 . \qquad (3.5)$$

The quantity I_C is the effective moment of inertia which is obtained in rotation about a horizontal axis passing through the center of inertia.

F 3 3

If a couple perpendicular to the cylindrical axis tending to rotate the cylinder about a horizontal axis is applied to the container, the container will rotate about an axis passing through the center of inertia. We thus see that as refers to the application of a horizontal couple, the center of inertia plays a role analogous to the center of gravity of a rigid body.

Let us now consider translational motion along the y axis. According to (2.19), the force and moment are

$$P_y = -m_1 \ddot{\gamma}, \qquad M_x = -\frac{h}{2} m_0 \ddot{\gamma} \qquad (3.6)$$

Let us write (3.6) in terms of the center of inertia. We then obtain the moment

$$M_{xC} = -\frac{1}{2} h m_0 \ddot{\gamma} - P_y L_C . \qquad (3.7)$$

Inserting the value of L_C from (3.4), we arrive at $M_x = 0$.

We thus see that when the container translates along a horizontal axis, the system of pressure force reduces to a horizontal force parallel to the axis of translation and passing through the center of inertia. This result means that if a horizontal force is applied to the container such that its line of action crosses the cylindrical axis at the center of inertia, the container will translate horizontally in the direction of the applied force similarly as would occur if a force were applied at the center of gravity of a rigid body.

Let us now consider a container to which is applied a set of forces parallel to the yz plane. Let us apply this set of forces to the center of inertia. Let R and N be the resultant and the moment of these forces, respectively. It follows from the above that under the action of R_y and R_z the container will undergo translation, and that under the action of the moment it will rotate about a horizontal axis passing through the center of inertia. If we write out the expressions which state that the liquid pressure forces on the walls are equal and opposite to the external forces, we obtain the following equations describing the motion of the center of inertia and the motion about the center of inertia:

$$m_1\ddot{\gamma} = R_y, \qquad m\ddot{\delta} = R_z, \qquad I_c\ddot{\beta} = N \tag{3.8}$$

where δ is the vertical displacement of the container in the z direction. From the external point of view, these equations are entirely analogous to similar ones for the motion of the center of gravity and the motion about the center of gravity of a rigid body.

The essential difference is that the inertia of the system is different for motion along the y and z directions (m_1 is not in general equal to m). In addition, the inertial characteristics and the position of the center of inertia will remain constant during the motion only for a certain class of motion, and they will be different for different motions within this class. Equations (3.8) can be used in treating the beginning of motion, impulsive forces, and steady state harmonic vibrations.

Let us write the expression for the moment of inertia in rotation about an arbitrary horizontal axis passing through the cylindrical axis. We have

$$\gamma = -\beta L.$$

The force and the moment about the x axis will be

$$P_y = m_1 L\ddot{\beta} - \tfrac{1}{2} h m_0 \ddot{\beta}, \qquad M_x = \tfrac{1}{2} hLm_0\ddot{\beta} - I_0\ddot{\beta}. \tag{3.9}$$

The moment about the axis of rotation will be

$$M_L = M_x - P_y L.$$

Inserting the values of M_x and P_y, we obtain

$$M_L = -I_L\ddot{\beta} \qquad (I_L = I_0 - m_0 hL + m_1 L^2)$$

where I_L is the effective moment of inertia for rotation about an axis at a distance L from the x axis.

Let l be the distance from the axis of rotation to the center of inertia, so that

$$L = L_c + l.$$

Inserting this into the equation for I_L, and making use of (3.5), we obtain

$$I_L = I_c + m_1 l^2 \tag{3.10}$$

Now (3.10) is analogous to the well known theorem of Steiner for the moment of inertia for a rigid body, and is a generalization of this theorem to the case of a liquid in a container. The second of (3.5) is a special case of this formula.

If $m_1 > 0$, then the moment of inertia I_L is minimal, as in the case of a solid body with $I = 0$, or about an axis passing through the center of inertia. For the case of impulsive forces, I_C and m_1 are always positive. For harmonic vibrations, there may be values of α for which $m_1 < 0$. For these cases the opposite situation occurs, that is I_L is a maximum for an axis passing through the center of inertia.

Let us now write out a formula for calculating the forces, moments and inertial characteristics under the coordinate transformation $x' = x$, $y' = y$, $z' = z + L$.

The expression for the force and moment for both displacement and rotation can be written .

$$P_{y'} = -m_1\ddot{\gamma}' - \frac{1}{2} h m_0'\ddot{\beta}, \qquad M_{x'} = -\frac{1}{2} h m_0'\ddot{\gamma}' - I_0'\ddot{\beta} \tag{3.11}$$

where γ' is the displacement with respect to the new origin O', and

$$m_0' = m_0 - \frac{2L}{h} m_1, \qquad I_0' = I_0 - m_0 h L + m_1 L^2 . \tag{3.12}$$

The effective mass m_1 is not changed by a shift of the origin along the $\underline{z}$ axis. The effective mass m_0 vanishes if the origin coincides with the center of inertia.

Together with the formulas for calculating m_0, m_1, I_0, Equations (3.11) and (3.12) can be used, for instance, to calculate the effective moments of inertia with respect to a transverse axis and the effective masses for a system containing several cylindrical cavities partially filled with liquid and lying along a single axis. It should be noted that for such systems it is also possible to define a center of inertia with similar properties. If a set of external forces is applied at this point, the inertial characteristic m_0 of the whole system will vanish, and the set of equations breaks up into independent equations which describe independently the motion of the center of inertia and the motion about the center of inertia. In this case the set of equations will be of the form of (3.8).

As for the case of space motion, if the weight is distributed with rotational symmetry, the set of equations for space motion in our case of small oscillations can be subdivided into two independent pairs of equations describing rotation and lateral motion for two axes perpendicular to the axis of symmetry, and equations for the translational motion along the symmetry axis and rotational motion about it. These two last equations are independent of each other and independent of the other equations of the system. The equation for motion along the symmetry axis will be the same as that for a system with a solidified liquid, and the equation of motion about the axis will be identical with the equation in the absence of liquid filling.

In the general case of arbitrary motion, the concept of inertial characteristics of a system with liquid filling becomes meaningless, and the first two pairs of equations are conveniently written in a somewhat different form in which they are treated, for instance, as the equations of motion of the solid part of the system with the pressure forces the liquid applies to the walls handled as external forces.

It is possible to add to the inertial characteristics of the solid part of the system that part of the inertial characteristics of the liquid which is independent of the wave motion, that is, independent of the χ_n. The terms containing these functions are then written out separately and treated as external forces.

A convenient form of the equations of motion is one in which the inertia of the system is treated as the sum of the inertial characteristics of the rigid part of the system and the solidified liquid, added to that part due to rotation and wave motion. In this case, if the center of application of the forces is treated as the center of gravity of the system with the solidified liquid, the principal terms are written in the same way as for a rigid body, and the equations obtained are coupled only because of the mobility of the liquid. These principal terms will always appear separately and can always be evaluated. The additional terms of the equations are obtained from the last terms of the initial characteristics which depend on the χ_n.

In order to complete each of two pairs of equations describing the motion in two perpendicular planes passing through the axis of symmetry (the momentum equation and the angular momentum equation), it is necessary to add to each of them an infinite set of Equations (1.28) which gives the relation between the wave motion parameters in each of the cylindrical cavities, if there are several of them, and the parameters of motion of the rigid part of the system.

In this case, the quantity y entering into the right side of these equations is understood not as the displacement of the center of application of the system, but as the displacement of the center of the free surface for each of the cavities.

4. Investigation of the inertial characteristics. In the motion of a container of liquid, m_1 and I_0 are of fundamental significance. The effective mass m_1 plays the role of the mass in lateral translation. The effective moment of inertia determines the inertia to rotation about the center of the free surface. The characteristic

m_ϕ does not have such clear physical meaning, and serves to determine the center of inertia.

Let us first consider the start of motion (impulsive forces).

Figure 6 gives the dependence of the quanties

$$m_1^\circ = \frac{m_1}{m}, \qquad m_\phi^\circ = \frac{m_\phi}{m}, \qquad I_\phi^\circ = \frac{I_\phi}{ma^2}$$

on the relative height h° of the container.

When h° = 0, we see that $m_1 = 0$.

As h° increases, so does m_1, approaching unity asymptotically.

This can be explained in the following way. The difference between m_1 and m is due to the fact that under a lateral force the liquid is not carried along by the container as a whole, lagging somewhat so that the liquid level drops somewhat on the forward wall and rises somewhat on the back one. For a small value of h° the walls hardly constrain the liquid to move and it falls behind strongly. Then the effective mass will be much less than the mass of the liquid. When the depth is great, on the other hand, the walls provide a strong constraint to the liquid motion and as the container is moved the liquid moves almost as a whole.

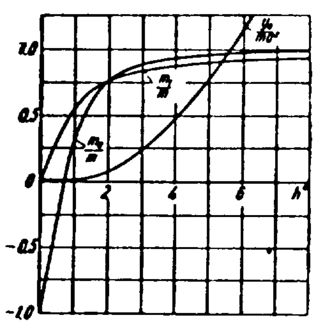

Fig. 6.

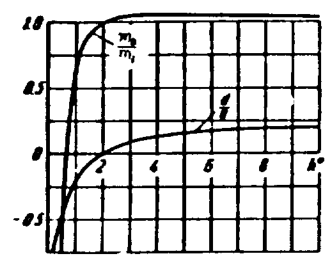

Fig. 7.

The lagging of the liquid, which takes place close to the free surface, leads only to very small displacements with respect to the container in the deeper layers of the liquid, since, as is known, the intensity of agitation drops very rapidly with depth.

Therefore for large values of h° the effective mass m_1 will hardly differ from the mass of the liquid.

Let us now discuss the manner in which the position of the center of inertia, as given by (3.4), depends on h°.

Figure 7 gives a graph of the ratio m_0/m_1. We see that as $h^* \to 0$, this ratio approaches $-\infty$. It can be shown, however, that $\frac{1}{2}h^* m_0/m_1$ remains finite and approaches a negative limit. This means that the center of inertia approaches a limiting position above the center of the free surface.

As h^* increases, so does the ratio m_0/m_1, and in the limit $h^* \to \infty$ this ratio approaches unity from above. This means that as h^* increases the center of inertia drops, crosses the free surface, and drops below the center of the container.

The distance from the center of inertia to the center of the container as given by (3.4) is

$$d = a\frac{h^*}{2}\left(\frac{m_0}{m_1} - 1\right). \tag{4.1}$$

A graph of d/a is shown in Fig. 7. We see that as $h^* \to \infty$ the distance between the center of inertia and the center of the container approaches a limit equal to

$$\lim d = a\sum_{n=1}^{\infty} \frac{1}{\zeta_n(\zeta_n^2 - 1)} \approx 0.23a \tag{4.2}$$

which means that in the limit the center of inertia lies somewhat less than a quarter of the radius below the center of the container. This ratio converges on its limit rather slowly. When $h^* = 7$, we have $d = 0.17a$.

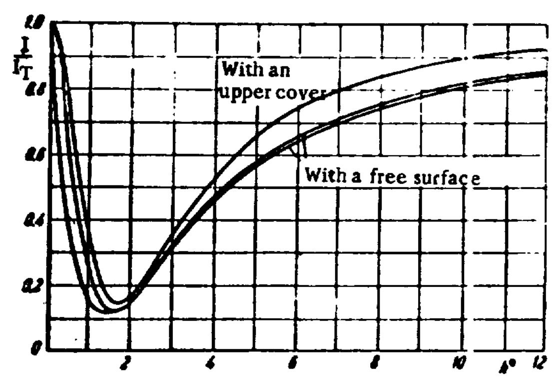

Fig. 8.

In order to discuss the role of rotation and wave motion, we have given in Fig. 8 the dependence on h^* of the ratio between the effective moment of inertia and the moment of inertia I_T of the solidified liquid, for a container with a cover on the free surface and one without it rotating about its center, and for a container with a free surface rotating about the center of inertia of the liquid.

We see that because of the rotation of the liquid the moment of inertia decreases very significantly for certain heights of the container in the case of a covered surface (by a factor of about 6 when $h^* = 1.5 - 1.8$). We see also that if

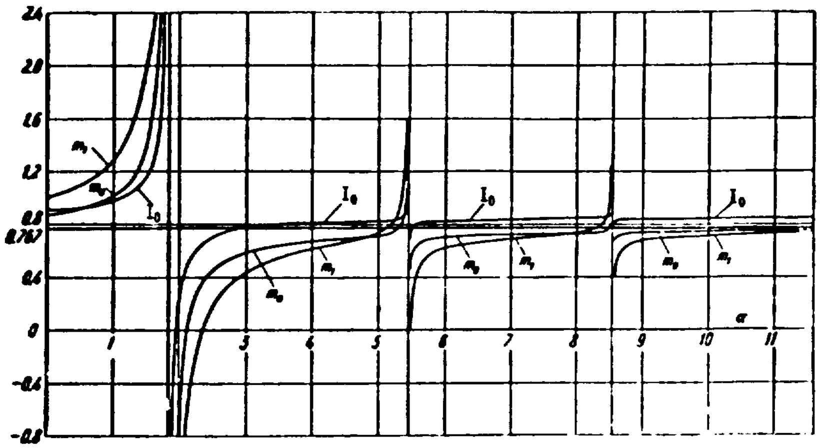

Fig. 9.

there is a free surface, the moment of inertia is further decreased. The influence of the free surface, however, is relatively weak, the rotation of the liquid playing the most important role. This is explained by the fact that in the case we are considering the axis of rotation is located within the cylinder. When the axis of rotation is moved further away, the influence of the free surface increases, and in the limit it becomes just as important as it is in translational motion.

The lower curve in Fig. 8 is for rotation about the center of inertia of the liquid in the container. At $h^* \approx 2$ the two lower curves are tangent (the center of inertia coincides with the center of the container). Since $m_1 > 0$, the moment of inertia is minimum about the center of inertia. A significant difference between the moment of inertia with respect to the center of the container occurs only for small values of h^*, when the center of inertia lies relatively far from the center of the container (Fig. 7).

Let us now consider the case of steady state harmonic vibrations. Figure 9 shows how the inertial characteristics m_1^*, m_0^*, I_0^* depend on α for $h^* = 2$. For $\alpha = 0$, we have $m_1^* = 1$, which means that the effective mass m_1 is equal to the mass of the liquid. This is clear, since the frequency of external vibrations is much less than the natural, and the liquid will therefore be displaced together with the container as a single unit. As α increases, the mass m_1 increases, becoming unbounded as the frequency of the first harmonic is approached. After the resonance m_1 is first negative, then passes through zero and becomes positive.

The unbounded increase of the effective mass close to resonance is convenient to think of not as the unbounded increase of the amplitude of vibration in the container, but as an indication that liquid vibrations will take place under very small amplitudes of vibration of the container. At the resonant frequency the container need not be displaced at all. This is understandable, as it is the frequency of vibration of the liquid in the stationary container.

Resonance will occur for all of the natural frequencies of vibration given by

$$\omega_n = \sqrt{\zeta_n \frac{g}{a} \operatorname{th}\left(\zeta_n \frac{h}{a}\right)}.$$

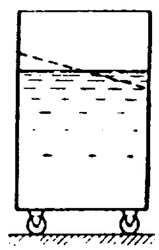

Fig. 10.

As $\alpha \to \infty$, the mass m_1 approaches a limit equal to its value for impulsive forces, and the width of the interval about the natural frequency where m_1 differs from its limiting value by more than a given quantity approaches zero. The limit for m_1^* is 0.767.

The effective mass is proportional to the negative of the horizontal projection of the resultant pressure force applied by the liquid on the wall of the container, or the external force which must be applied to the container in order to obtain oscillations with the given frequency. When the effective mass vanishes, this means that at the given frequency the container can oscillate without the application of horizontal external forces. These frequencies ω_n' form in infinite series. and are the roots of equations which we obtain when we set $m_1 = 0$ in Equation (2.26). The frequency ω_n' is the frequency of free vibrations of the liquid together with a weightless container for a container which undergoes translation (Fig. 10). In all cases $\omega_n' > \omega_n$, and $\omega_n' \to \omega_n$ as $n \to \infty$.

The positive or negative signs on the effective mass m_1 indicate that in order to maintain oscillations at a given frequency the force applied to the liquid must be directed parallel to the acceleration or antiparallel to it. In other words, the force and acceleration must have the same or opposite phases.

Oscillations of a liquid with a negative mass m_1 are obtained, for instance, when we consider the natural vibrations of a liquid together with a massive container. We have $M + m_1 = 0$, where M is the mass of the container. From this we have

$$m_1 = -M < 0\,.$$

The natural frequencies are obtained from Equation (2.26) by setting $m_1 = -M$. The roots ω_n° of this equation satisfy the physically obvious condition

$$\omega_n < \omega_n^{\circ} < \omega_n'\,.$$

We mention also the physical meaning of allowing the center of inertia to go to infinity. This occurs at frequencies for which $m_1 = 0$, or when the force necessary to maintain oscillations of the container vanishes. The necessary moment in this case, however, does not vanish. This moment is produced by the constraints which cause the cylinder to translate.

The graphs of Fig. 9 show that the moment of inertia I_0 and the moment of inertia for rotation about an arbitrary horizontal axis depend similarly on α. The physical meaning of a vanishing, positive, or negative moment of inertia is the same as that for the effective mass m_1. There is some difference due to the moment of the hydrostatic forces. When the moment of inertia vanishes, for instance, the corresponding frequency is therefore not equal to the frequency of oscillation of the liquid together with a weightless container, but takes on this value only if the axis of rotation passes through the center of inertia, and if the moment of the hydrostatic forces vanishes.

The above investigation shows that the presence of a free surface may in many cases lead to measurable differences in the inertial characteristics of bodies with liquid-filled cavities.

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